

ON THE EXISTENCE OF TANGENTIAL HOLOMORPHIC VECTOR FIELDS VANISHING AT AN INFINITE TYPE POINT

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ABSTRACT. The purpose of this article is to investigate the holomorphic vector fields tangent to a real hypersurface in \mathbb{C}^2 vanishing at an infinite type point.

1. INTRODUCTION

A holomorphic vector field in \mathbb{C}^n takes the form

$$H = \sum_{k=1}^n h_k(z) \frac{\partial}{\partial z_k}$$

for some functions h_1, \dots, h_n holomorphic in $z = (z_1, \dots, z_n)$. A smooth real hypersurface germ M (of real codimension 1) at p in \mathbb{C}^n takes a defining function, say ρ , such that M is represented by the equation $\rho(z) = 0$. The holomorphic vector field H is said to be *tangent* to M if its real part $\operatorname{Re} H$ is tangent to M , i.e., H satisfies the equation $\operatorname{Re} H \rho = 0$.

In several complex variables, such tangential holomorphic vector fields arise naturally from the action by the automorphism group of a domain. Analysis of such vector fields has turned out to be quite crucial: cf., e.g., [1, 2, 3] in which the existence of parabolic vector fields plays an important role. Moreover, the study of tangential holomorphic vector fields pertains to the Greene-Krantz's conjecture, that is, for a smoothly bounded pseudoconvex domain admitting a non-compact automorphism group, orbits can accumulate only at a point of D'Angelo finite type [7]. For this conjecture, we refer readers to the recent papers [8, 9] and the references therein.

This paper continues the work that started in [9] motivated by the following question.

Problem. Assume that (M, p) is a non-Leviflat CR hypersurface germ in \mathbb{C}^n such that p is a point of D'Angelo infinite type. Characterize all holomorphic vector fields tangent to M vanishing at p .

More precisely, we present a characterization of holomorphic vector fields which are tangent to a C^∞ -smooth hypersurface germ $(M, 0)$ of D'Angelo infinite type at the origin $0 = (0, 0)$ in \mathbb{C}^2 and vanish at 0.

Let M be such a C^∞ -smooth real hypersurface germ $(M, 0)$. Then it admits the following expression.

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + (\operatorname{Im} z_1)Q(z_2, \operatorname{Im} z_1) = 0\}, \quad (1)$$

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where P and Q are C^∞ -smooth functions with $P(0) = 0$, $dP(0) = 0$, and $Q(0, 0) = 0$. We note that if P contains no harmonic terms, then M is of D'Angelo infinite type if and only if P vanishes to infinite order at 0 (see [9, Theorem 2]).

In the case that $P(z_2)$ is positive on a punctured disk, K.-T. Kim and the author [9] showed that there is no non-trivial holomorphic vector field vanishing at the origin tangent to any C^∞ -smooth real hypersurface germ $(M, 0)$, except the two following cases:

- (i) The vanishing order of $Q(z_2, 0)$ at $z_2 = 0$ is finite and $Q(z_2, 0)$ contains a monomial term z_2^k for some positive integer k .
- (ii) The real hypersurface M is rotationally symmetric, i.e. $\rho(z_1, z_2) = \rho(z_1, |z_2|)$, and in this case the holomorphic vector field is of the form $i\beta z_2 \frac{\partial}{\partial z_2}$ for some non-zero real number β (see also [4]).

In addition, an example for the exceptional case (i) was also given.

The first aim of this paper is to prove the following theorem, which gives a classification of pairs (H, M) of holomorphic vector fields H tangent to real hypersurfaces M .

Theorem 1. *If a non-trivial holomorphic vector field germ $(H, 0)$ vanishing at the origin is tangent to a real non-rotationally symmetric hypersurface germ $(M, 0)$ defined by the equation $\rho(z) := \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + f(z_2, \operatorname{Im} z_1) = 0$ satisfying the conditions:*

- (1) $f(z_2, t)$ is real analytic in a neighborhood of $0 \in \mathbb{C} \times \mathbb{R}$ satisfying $f(z_2, 0) \equiv 0$,
- (2) $P(z_2) > 0$ for any $z_2 \neq 0$,
- (3) P vanishes to infinite order at $z_2 = 0$, and
- (4) There is no positive integer number n such that $\log(P(z_2)) \approx -\frac{1}{|z_2|^n}$,

then, after a change of variable in z_2 , there exist $\alpha, \beta, \epsilon_0, \delta_0 \in \mathbb{R}$ with $\beta \neq 0, \epsilon_0 > 0, \delta_0 > 0$ such that

$$H(z_1, z_2) = L(z_1)a_1(z_2)\frac{\partial}{\partial z_1} + i\beta z_2\frac{\partial}{\partial z_2},$$

where $a_1(z_2) = \beta \sum_{n=1}^{\infty} a_n z_2^n$ ($a_n \in \mathbb{C}$, $n \geq 1$) is holomorphic in $\Delta_{\epsilon_0} := \{z_2 \in \mathbb{C} : |z_2| < \epsilon_0\}$ and

$$L(z_1) = \begin{cases} z_1 & \text{if } \alpha = 0 \\ \frac{1}{\alpha} (\exp(\alpha z_1) - 1) & \text{if } \alpha \neq 0, \end{cases}$$

and f and P are respectively defined on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ and Δ_{ϵ_0} by

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where $R(z_2) = q(|z_2|) - \operatorname{Re}(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n)$ for all $z_2 \in \Delta_{\epsilon_0}$, and

$$P(z_2) = \begin{cases} \frac{1}{\alpha} \log [1 + \alpha P_1(z_2)] & \text{if } \alpha \neq 0 \\ P_1(z_2) & \text{if } \alpha = 0, \end{cases}$$

where

$$P_1(z_2) = \exp \left[p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log |\cos(R(z_2))| \right]$$

for all $z_2 \in \Delta_{\epsilon_0}^*$ and $P_1(0) = 0$, and q, p are reasonable functions defined on $[0, \epsilon_0)$ and $(0, \epsilon_0)$ respectively with $q(0) = 0$, e.g. $q(t) = 0$ and $p(t) = -\frac{1}{t^\alpha}$ ($\alpha > 0$) for all $t > 0$, so that P is C^∞ -smooth on Δ_{ϵ_0} and vanishes to infinite order at $z_2 = 0$ and R is real analytic in Δ_{ϵ_0} .

Here and in what follows, \lesssim and \gtrsim denote inequalities up to a positive constant. In addition, we use \approx for the combination of \lesssim and \gtrsim .

Remark 1. As to the hypothesis of the theorem, the condition (3) simply says that 0 is a point of D'Angelo infinite type. The last condition (4) is the only technical condition but is crucial to conclude that the coefficient of $\frac{\partial}{\partial z_2}$ is $i\beta z_2$.

We note that the holomorphic vector field and the real hypersurface given in Theorem 1 are tangent. Moreover, general examples are given (cf. Theorem 3 in Sec. 3).

We would like to emphasize here that the assumption on the positiveness of a function P is essential in the proofs of Theorem 1 and the main theorem in [9]. The following result, in which the positivity of a function P is not necessary, is our second main result of this article.

Theorem 2. *If a C^∞ -smooth hypersurface germ $(M, 0)$ is defined by the equation $\rho(z) := \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + (\operatorname{Im} z_1)Q(z_2, \operatorname{Im} z_1) = 0$, satisfying the conditions:*

- (1) $P \not\equiv 0$, $P(0) = 0$;
- (2) P satisfies the condition (I) (cf. Definition 1 in Sec. 2);
- (3) P vanishes to infinite order at $z_2 = 0$,

then any holomorphic vector field vanishing at the origin tangent to $(M, 0)$ is identically zero.

Remark 2 (Notations). Taking the risk of confusion we employ the notation

$$P'(z) = P_z(z) = \frac{\partial P}{\partial z}(z); f_z(z, t) = \frac{\partial f}{\partial z}(z, t); f_t(z, t) = \frac{\partial f}{\partial t}(z, t)$$

throughout the paper. Of course for a function of single real variable $f(t)$, we shall continue using $f'(t)$ for its derivative, as well.

Following [5], we consider a smooth real-valued function f defined in a neighborhood of 0 in \mathbb{C} . Let $\nu_0(f)$ denote the order of vanishing of f at 0, by the first nonvanishing degree term in its Taylor expansion at 0. In case f is a mapping into \mathbb{R}^k ($k > 1$), we consider the order of vanishing of all the components and take the smallest one among them for the vanishing order of f . Denote it by $\nu_0(f)$. Also denote by $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ for $r > 0$ and by $\Delta := \Delta_1$. Then the origin is called a *point of D'Angelo infinite type* if, for every integer $\ell > 0$, there exists a holomorphic map $h : \Delta \rightarrow \mathbb{C}^2$ with $h(0) = (0, 0)$ such that

$$\nu_0(h) \neq \infty \text{ and } \frac{\nu_0(\rho \circ h)}{\nu_0(h)} > \ell.$$

This paper is organized as follows. In Section 2, we introduce the condition (I) and give several examples of functions satisfying the condition (I). In Section 3, we prove Theorem 3. Section 4 is devoted to the proof of Theorem 1. The proof of Theorem 2 is given in Section 5. Finally, several technical lemmas are pointed out in Appendix A.

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2. FUNCTIONS VANISHING TO INFINITE ORDER

In this section, we will introduce the condition (I) and give several examples of functions defined on the open unit disc in the complex plane with infinite order of vanishing at the origin.

Definition 1. We say that a real smooth function f defined on a neighborhood U of the origin in \mathbb{C} satisfies the *condition (I)* if

$$(I.1) \quad \limsup_{\tilde{U} \ni z \rightarrow 0} |\operatorname{Re}(bz^k \frac{f'(z)}{f(z)})| = +\infty;$$

$$(I.2) \quad \limsup_{\tilde{U} \ni z \rightarrow 0} |\frac{f'(z)}{f(z)}| = +\infty,$$

for all $k = 1, 2, \dots$ and for all $b \in \mathbb{C}^*$, where $\tilde{U} := \{z \in U : f(z) \neq 0\}$.

Example 1. The function $P(z) = e^{-C/|\operatorname{Re}(z)|^\alpha}$ if $\operatorname{Re}(z) \neq 0$ and $P(z) = 0$ if otherwise, where $C, \alpha > 0$, satisfies the condition (I). Indeed, a direct computation shows that

$$P'(z) = P(z) \frac{C\alpha}{2|\operatorname{Re}(z)|^{\alpha+1}}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re}(z) \neq 0$. Therefore, it is easy to see that $|P'(z)/P(z)| \rightarrow +\infty$ as $z \rightarrow 0$ with $\operatorname{Re}(z) \neq 0$. We shall prove that the conditions (I.1) and (I.2) hold. Let k be an arbitrary positive integer. Let $z_l := 1/l + i/l^\beta$, where $0 < \beta < \min\{1, \alpha/(k-1)\}$ if $k > 1$ and $\beta = 1/2$ if $k = 1$, for all $l \in \mathbb{N}^*$. Then $z_l \rightarrow 0$ as $l \rightarrow \infty$ and $\operatorname{Re}(z_l) = 1/l \neq 0$ for all $l \in \mathbb{N}^*$. Moreover, for each $b \in \mathbb{C}^*$ we have that

$$|\operatorname{Re}(bz_l^k \frac{P'(z_l)}{P(z_l)})| \gtrsim \frac{l^{\alpha+1}}{l^{\beta(k-1)+1}} = l^{\alpha-\beta(k-1)}.$$

This implies that

$$\lim_{l \rightarrow \infty} |\operatorname{Re}(bz_l^k \frac{P'(z_l)}{P(z_l)})| = +\infty.$$

Hence, the function P satisfies the condition (I).

Remark 3. i) Any rotational function P does not satisfy the condition (I.1) because $\operatorname{Re}(izP'(z)) = 0$ (see [9] or [4]).

ii) It follows from [9, Lemma 2] that if P is a non-zero function defined on a neighborhood U of the origin in \mathbb{C} and $\tilde{U} := \{z \in U : P(z) \neq 0\}$ contains a \mathcal{C}^1 -smooth curve $\gamma : (0, 1] \rightarrow \tilde{U}$ such that γ' stays bounded on $(0, 1]$ and $\lim_{t \rightarrow 0^+} \gamma(t) = 0$, then P satisfies the condition (I.2).

Lemma 1. Suppose that $g : (0, 1] \rightarrow \mathbb{R}$ is a \mathcal{C}^1 -smooth unbounded function. Then we have $\limsup_{t \rightarrow 0^+} t^\alpha |g'(t)| = +\infty$ for any real number $\alpha < 1$.

Proof. Fix an arbitrary $\alpha < 1$. Suppose that, on the contrary, $\limsup_{t \rightarrow 0^+} t^\alpha |g'(t)| < +\infty$. Then there is a constant $C > 0$ such that

$$|g'(t)| \leq \frac{C}{t^\alpha}, \quad \forall 0 < t < 1.$$

We now have the following estimate

$$\begin{aligned} |g(t)| &\leq |g(1)| + \int_t^1 |g'(\tau)| d\tau \leq |g(1)| + C \int_t^1 \frac{d\tau}{\tau^\alpha} \\ &\leq |g(1)| + \frac{C}{1-\alpha} (1 - t^{1-\alpha}) \lesssim 1. \end{aligned}$$

However, this is impossible since g is unbounded on $(0, 1]$, and thus the lemma is proved. \square

In general, the above lemma does not hold for $\alpha \geq 1$. This follows from that $|t^{1+\beta} \frac{d}{dt} \frac{1}{t^\beta}| = \beta$ and $|t \frac{d}{dt} \log(t)| = 1$ for all $0 < t < 1$, where $\beta > 0$. However, the following lemmas show that there exists such a function g such that $\liminf_{t \rightarrow 0^+} \sqrt{t}|g'(t)| < +\infty$ and $\limsup_{t \rightarrow 0^+} t^\beta |g'(t)| = +\infty$ for all $\beta < 2$. Furthermore, several examples of smooth functions vanishing to infinite order at the origin in \mathbb{C} and satisfying the condition (I) are constructed.

Lemma 2. *There exists a \mathcal{C}^∞ -smooth real-valued function $g : (0, 1) \rightarrow \mathbb{R}$ satisfying*

- (i) $g(t) \equiv -2n$ on the closed interval $\left[\frac{1}{n+1} \left(1 + \frac{1}{3n}\right), \frac{1}{n+1} \left(1 + \frac{2}{3n}\right)\right]$ for $n = 4, 5, \dots$;
- (ii) $g(t) \approx \frac{-1}{t}, \forall t \in (0, 1)$;
- (iii) for each $k \in \mathbb{N}$ there exists $C(k) > 0$, depending only on k , such that $|g^{(k)}(t)| \leq \frac{C(k)}{t^{3k+1}}, \forall t \in (0, 1)$.

Remark 4. Let

$$P(z) := \begin{cases} \exp(g(|z|^2)) & \text{if } 0 < |z| < 1 \\ 0 & \text{if } z = 0. \end{cases}$$

Then this function is a \mathcal{C}^∞ -smooth function on the open unit disc Δ that vanishes to infinite order at the origin. Moreover, we see that $P'(\frac{2n+1}{2n(n+1)}) = 0$ for any $n \geq 4$, and hence $\liminf_{z \rightarrow 0} |P'(z)|/P(z) = 0$.

Lemma 2 was stated in [9] without proof. A detailed proof of this lemma is given in Appendix A.1.

Lemma 3. *Let $h : (0, +\infty) \rightarrow \mathbb{R}$ be the piecewise linear function such that $h(a_n) = h(b_n) = 2^{2 \cdot 4^{n-1}}$, $h(1/2) = \sqrt{2}$ and $h(t) = 0$ if $t \geq 1$, where $a_n = 1/2^{4^n}$, $a_0 = 1/2$, $b_n = (a_n + a_{n-1})/2$ for every $n \in \mathbb{N}^*$. Then the function $f : (0, 1) \rightarrow \mathbb{R}$ given by*

$$f(t) = - \int_t^1 h(\tau) d\tau$$

satisfies:

- (i) $f'(a_n) = \frac{1}{\sqrt{a_n}}$ for every $n \in \mathbb{N}^*$;
- (ii) $f'(b_n) \sim \frac{1}{4b_n^2}$ as $n \rightarrow \infty$;
- (iii) $-\frac{1}{t} \lesssim f(t) \lesssim -\frac{1}{t^{1/16}}, \forall 0 < t < 1$.

Proof. We have $f'(a_n) = h(a_n) = 2^{2 \cdot 4^{n-1}} = \frac{1}{\sqrt{a_n}}$, which proves (i). Since $b_n = (a_n + a_{n-1})/2 \sim a_{n-1}/2$ as $n \rightarrow \infty$, we have $f'(b_n) = h(b_n) = 2^{2 \cdot 4^{n-1}} = \frac{1}{a_{n-1}^2} \sim \frac{1}{4b_n^2}$

as $n \rightarrow \infty$. So, the assertion (ii) follows. Now we shall show (iii). For an arbitrary real number $t \in (0, 1/16)$, denote by N the positive integer such that

$$1/2^{4^{N+1}} \leq t < 1/2^{4^N}.$$

Then it is easy to show that

$$\begin{aligned} f(t) &\leq -\int_{a_N}^{b_N} h(\tau) d\tau = -\frac{1}{2} 2^{2 \cdot 4^{N-1}} (1/2^{4^{N-1}} - 1/2^{4^N}) \\ &\leq -\frac{1}{2} 2^{4^{N-1}} + \frac{1}{8} \leq -\frac{1}{2} \frac{1}{t^{1/16}} + \frac{1}{8} \lesssim -\frac{1}{t^{1/16}}; \\ f(t) &\geq -2 \int_{a_{N+1}}^{b_{N+1}} h(\tau) d\tau - \int_{a_N}^1 h(\tau) d\tau \\ &\geq -2h(a_{N+1})(b_{N+1} - a_{N+1}) - h(a_N)(1 - a_N) \\ &\geq -2^{2 \cdot 4^N} (1/2^{4^N} - 1/2^{4^{N+1}}) - 2^{2 \cdot 4^{N-1}} (1 - 1/2^{4^N}) \\ &\gtrsim -\frac{1}{t} \end{aligned}$$

for any $0 < t < 1/16$. Thus (iii) is shown. \square

Remark 5. i) We note that f is \mathcal{C}^1 -smooth, increasing, and concave on the interval $(0, 1)$. By taking a suitable regularization of the function f as in the proof of Lemma 2, we may assume that it is \mathcal{C}^∞ -smooth and still satisfies the above properties (i), (ii), and (iii). In addition, for each $k \in \mathbb{N}$ there exist $C(k) > 0$ and $d(k) > 0$, depending only on k , such that $|f^{(k)}(t)| \leq \frac{C(k)}{t^{d(k)}}$, $\forall t \in (0, 1)$. Thus the function $R(z)$ defined by

$$R(z) := \begin{cases} \exp(f(|z|^2)) & \text{if } 0 < |z| < 1 \\ 0 & \text{if } z = 0 \end{cases}$$

is \mathcal{C}^∞ -smooth and vanishes to infinite order at the origin. Moreover, we have $\liminf_{z \rightarrow 0} |R'(z)/R(z)| < +\infty$ and $\limsup_{z \rightarrow 0} |R'(z)/R(z)| = +\infty$.

ii) Since the functions P, R are rotational, they do not satisfy the condition (I) (cf. Remark 3). On the other hand, the functions $\tilde{P}(z) := P(\operatorname{Re}(z))$ and $\tilde{R}(z) := R(\operatorname{Re}(z))$ satisfy the condition (I). Indeed, a simple calculation shows

$$\tilde{R}'(z) = \tilde{R}(z) f'(|\operatorname{Re}(z)|^2) \operatorname{Re}(z)$$

for any $z \in \mathbb{C}$ with $|\operatorname{Re}(z)| < 1$. By the above property (ii), it follows that $\limsup_{z \rightarrow 0} |\tilde{R}'(z)/\tilde{R}(z)| = +\infty$. Moreover, for each $k \in \mathbb{N}^*$ and each $b \in \mathbb{C}^*$ if we choose a sequence $\{z_n\}$ with $z_n := \sqrt{b_n} + i(\sqrt{b_n})^\beta$, where $0 < \beta < \min\{1, 2/(k-1)\}$ if $k > 1$ and $\beta = 1/2$ if $k = 1$, then $z_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$|\operatorname{Re}\left(b z_n^k \frac{\tilde{R}'(z_n)}{\tilde{R}(z_n)}\right)| \gtrsim \frac{(\sqrt{b_n})^{(k-1)\beta+2}}{b_n^2} \rightarrow +\infty$$

as $n \rightarrow \infty$. Hence, \tilde{R} satisfies the condition (I). Now it follows from the construction of the function g in the proof of Lemma 2 (cf. Appendix A.1) that $g'(\frac{1}{n}) \sim 3n^2$ as $n \rightarrow \infty$. Therefore, using the same argument as above we conclude that \tilde{P} also satisfies the condition (I).

It is not hard to show that the above functions such as $P, R, \tilde{P}, \tilde{R}$ are not subharmonic. Up to now it is unknown that there exists a \mathcal{C}^∞ -smooth subharmonic function P defined on the unit disc such that $\nu_0(P) = +\infty$ and $\liminf_{z \rightarrow 0} |P'(z)/P(z)| < +\infty$.

3. EXISTENCE OF HOLOMORPHIC VECTOR FIELDS TANGENT TO REAL HYPERSURFACES

Let $b(z) = i\beta z + \dots$ ($\beta \in \mathbb{R}^*$) be a holomorphic function on a neighborhood U of the origin. It was proved in [6] that there exists a conformal function $\Phi : V \rightarrow U$, where U and V are two open neighborhoods of the origin, such that $\Phi(0) = 0$ and $z(t) = \Phi(w_0 e^{i\beta t})$, $-\infty < t < +\infty$, is the solution of the differential equation $\frac{dz(t)}{dt} = b(z(t)) = i\beta z(t) + \dots$ satisfying $z(0) = \Phi(w_0) \in U$. Moreover, one gets

$$\Phi'(w)i\beta w = b(\Phi(w)) \text{ for all } w \in V.$$

The following lemma that will be of use later is a change of variables.

Lemma 4. *Let a, b be two holomorphic functions defined on neighborhoods $\Delta_r \times U$ and U of the origins in \mathbb{C}^2 and in \mathbb{C} , respectively, with $b(0) = 0$ and $b'(0) = i\beta$, where $\beta \in \mathbb{R}^*$ and $r > 0$. Then, after the change of variables*

$$z_1 = w_1; z_2 = \Phi(w_2),$$

we obtain that

$$H(z_1, z_2) = a(z_1, z_2) \frac{\partial}{\partial z_1} + b(z_2) \frac{\partial}{\partial z_2}$$

is tangent to the hypersurface

$$M = \{(z_1, z_2) \in \Delta_r \times U : \rho(z_1, z_2) = \operatorname{Re} z_1 + f(z_2, \operatorname{Im} z_1) = 0\}$$

if and only if

$$\tilde{H}(w_1, w_2) = a(w_1, \Phi(w_2)) \frac{\partial}{\partial z_1} + i\beta w_2 \frac{\partial}{\partial z_2}$$

is tangent to the hypersurface

$$\tilde{M} = \{(w_1, w_2) \in \Delta_r \times V : \tilde{\rho}(w_1, w_2) = \operatorname{Re} w_1 + f(\Phi(w_2), \operatorname{Im} w_1) = 0\}.$$

Proof. Since $\Phi'(w_2)i\beta w_2 = b(\Phi(w_2))$ for all $w_2 \in V$, it follows that

$$i\beta w_2 f_{w_2}(\Phi(w_2), \operatorname{Im} w_1) = i\beta w_2 \Phi'(w_2) f_{z_2}(z_2, \operatorname{Im} z_1) = b(z_2) f_{z_2}(z_2, \operatorname{Im} z_1)$$

for all $w_2 \in V$. Therefore, we obtain that

$$\begin{aligned} \operatorname{Re} H(\rho(z_1, z_2)) &= \operatorname{Re} \left[\left(\frac{1}{2} + f_{z_1}(z_2, \operatorname{Im} z_1) \right) a(z_1, z_2) + f_{z_2}(z_2, \operatorname{Im} z_1) b(z_2) \right] \\ &= \operatorname{Re} \left[\left(\frac{1}{2} + f_{w_1}(\Phi(w_2), \operatorname{Im} w_1) \right) a(w_1, \Phi(w_2)) + f_{w_2}(\Phi(w_2), \operatorname{Im} w_1) i\beta w_2 \right] \\ &= \operatorname{Re} \tilde{H}(\tilde{\rho}(w_1, w_2)) \end{aligned}$$

for every $(w_1, w_2) \in \Delta_r \times V$, which proves the assertion. \square

The following theorem gives examples of holomorphic vector fields and real hypersurfaces which are tangent.

Theorem 3. Let $\alpha \in \mathbb{R}$ and let $a_1(z) = \beta \sum_{n=1}^{\infty} a_n z^n$ be a non-zero holomorphic function defined on a neighborhood of $0 \in \mathbb{C}$, where $\beta \in \mathbb{R}^*$ and $a_n \in \mathbb{C}$ for all $n \geq 1$. Then there exist positive numbers $\epsilon_0, \delta_0 > 0$ such that the holomorphic vector field

$$H(z_1, z_2) = L(z_1)a_1(z_2)\frac{\partial}{\partial z_1} + i\beta z_2\frac{\partial}{\partial z_2},$$

where

$$L(z_1) = \begin{cases} z_1 & \text{if } \alpha = 0 \\ \frac{1}{\alpha}(\exp(\alpha z_1) - 1) & \text{if } \alpha \neq 0, \end{cases}$$

is tangent to the \mathcal{C}^1 -smooth hypersurface M given by

$$M = \{(z_1, z_2) \in \Delta_{\delta_0} \times \Delta_{\epsilon_0} : \rho(z_1, z_2) := \operatorname{Re} z_1 + P(z_2) + f(z_2, \operatorname{Im} z_1) = 0\},$$

where f and P are respectively defined on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ and Δ_{ϵ_0} by

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where $R(z_2) = q(|z_2|) - \operatorname{Re}(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n)$ for all $z_2 \in \Delta_{\epsilon_0}$, and

$$P(z_2) = \begin{cases} \frac{1}{\alpha} \log [1 + \alpha P_1(z_2)] & \text{if } \alpha \neq 0 \\ P_1(z_2) & \text{if } \alpha = 0, \end{cases}$$

where

$$P_1(z_2) = \exp \left[p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log |\cos(R(z_2))| \right]$$

for all $z_2 \in \Delta_{\epsilon_0}^*$ and $P_1(0) = 0$, and q, p are reasonable functions defined on $[0, \epsilon_0)$ and $(0, \epsilon_0)$ respectively with $q(0) = 0$, e.g. $q(t) = 0$ and $p(t) = -\frac{1}{t^\alpha}$ ($\alpha > 0$) for all $t > 0$, so that P, R are \mathcal{C}^1 -smooth in Δ_{ϵ_0} .

Proof. First of all, it is easy to show that there is $\epsilon_0 > 0$ such that we can choose a function q so that the function R defined as in the theorem is \mathcal{C}^1 -smooth and $|R(z_2)| \leq 1$ on Δ_{ϵ_0} . Choose $\delta_0 = \frac{1}{2|\alpha|}$ if $\alpha \neq 0$ and $\delta_0 = +\infty$ if otherwise. Then the function $f(z_2, t)$ given in the theorem is \mathcal{C}^1 -smooth on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$. Moreover, $f(z_2, t)$ is real analytic in t and $\frac{\partial^m f}{\partial t^m}$ is \mathcal{C}^1 -smooth on $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ for each $m \in \mathbb{N}$.

Next, let P_1, P, R be functions defined as in the theorem and let $Q_0(z_2) := \tan(R(z_2))$ for all $z_2 \in \Delta_{\epsilon_0}$. Then by Lemma 7, Lemma 8, and Corollary 9 we have the following equations.

- (i) $\operatorname{Re} \left[i\beta z_2 Q_0(z_2) + \frac{1}{2} (1 + Q_0^2(z_2)) i a_1(z_2) \right] \equiv 0;$
- (ii) $\operatorname{Re} \left[i\beta z_2 P_1(z_2) - \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a_1(z_2) P_1(z_2) \right] \equiv 0;$
- (iii) $\operatorname{Re} \left[i\beta z_2 P(z_2) + \frac{\exp(-\alpha P(z_2)) - 1}{\alpha} \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a_1(z_2) \right] \equiv 0$ for $\alpha \neq 0;$
- (iv) $(i + f_t(z_2, t)) \exp(\alpha(it - f(z_2, t))) \equiv i + Q_0(z_2);$
- (v) $\operatorname{Re} \left[2i\alpha\beta z_2 f_{z_2}(z_2, t) + (f_t(z_2, t) - Q_0(z_2)) i a_1(z_2) \right] \equiv 0$

on Δ_{ϵ_0} for any $t \in (-\delta_0, \delta_0)$.

We now prove that the holomorphic vector field H is tangent to the hypersurface M . Indeed, by a calculation we get $\rho_{z_1}(z_1, z_2) = \frac{1}{2} + \frac{f_t(z_2, \text{Im } z_1)}{2i}$ and $\rho_{z_2}(z_1, z_2) = P_{z_2}(z_2) + f_{z_2}(z_2, \text{Im } z_1)$.

We devide the proof into two cases.

a) $\alpha = 0$. In this case, $f(z_2, t) = Q_0(z_2)t$ for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$. Therefore, by (i) and (ii) one obtains that

$$\begin{aligned} \text{Re } H(\rho(z_1, z_2)) &= \text{Re} \left[\left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) z_1 a_1(z_2) + \left(P_{1z_2}(z_2) + (\text{Im } z_1) Q_{0z_2}(z_2) \right) i\beta z_2 \right] \\ &= \text{Re} \left[\left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \left(i(\text{Im } z_1) - P_1(z_2) - (\text{Im } z_1) Q_0(z_2) \right) a_1(z_2) \right. \\ &\quad \left. + \left(P_{1z_2}(z_2) + (\text{Im } z_1) Q_{0z_2}(z_2) \right) i\beta z_2 \right] \\ &= \text{Re} \left[i\beta z_2 P_{1z_2}(z_2) - \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) a_1(z_2) P_1(z_2) \right] \\ &\quad + (\text{Im } z_1) \text{Re} \left[i\beta z_2 Q_{0z_2}(z_2) + \frac{1}{2} (1 + Q_0(z_2)^2) i a_1(z_2) \right] = 0 \end{aligned}$$

for every $(z_1, z_2) \in M$, which proves the theorem for $\alpha = 0$.

b) $\alpha \neq 0$. It follows from (iii), (iv), and (v) that

$$\begin{aligned} \text{Re } H(\rho(z_1, z_2)) &= \text{Re} \left[\left(\frac{1}{2} + \frac{f_t(z_2, \text{Im } z_1)}{2i} \right) L(z_1) a_1(z_2) + \left(P_{z_2}(z_2) + f_{z_2}(z_2, \text{Im } z_1) \right) i\beta z_2 \right] \\ &= \text{Re} \left[\left(\frac{1}{2} + \frac{f_t(z_2, \text{Im } z_1)}{2i} \right) \frac{1}{\alpha} \left(\exp \left(\alpha (i \text{Im } z_1 - P(z_2) - f(z_2, \text{Im } z_1)) \right) - 1 \right) a_1(z_2) \right. \\ &\quad \left. + \left(P_{z_2}(z_2) + f_{z_2}(z_2, \text{Im } z_1) \right) i\beta z_2 \right] \\ &= \text{Re} \left[\frac{1}{\alpha} \frac{i + f_t(z_2, \text{Im } z_1)}{2i} \exp \left(\alpha (i \text{Im } z_1 - f(z_2, \text{Im } z_1)) \right) \exp(-\alpha P(z_2)) a_1(z_2) \right. \\ &\quad \left. - \frac{1}{\alpha} \left(\frac{1}{2} + \frac{f_t(z_2, \text{Im } z_1)}{2i} \right) a_1(z_2) + \left(P_{z_2}(z_2) + f_{z_2}(z_2, \text{Im } z_1) \right) i\beta z_2 \right] \\ &= \text{Re} \left[\frac{1}{\alpha} \frac{i + Q_0(z_2)}{2i} \exp(-\alpha P(z_2)) a_1(z_2) - \frac{1}{\alpha} \left(\frac{1}{2} + \frac{f_t(z_2, \text{Im } z_1)}{2i} \right) a_1(z_2) \right. \\ &\quad \left. + \left(P_{z_2}(z_2) + f_{z_2}(z_2, \text{Im } z_1) \right) i\beta z_2 \right] \\ &= \text{Re} \left[i\beta z_2 P_{z_2}(z_2) + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \frac{\exp(-\alpha P(z_2)) - 1}{\alpha} a_1(z_2) \right] \\ &\quad + \text{Re} \left[i\beta z_2 f_{z_2}(z_2, \text{Im } z_1) + \frac{1}{2\alpha} \left(f_t(z_2, \text{Im } z_1) - Q_0(z_2) \right) i a_1(z_2) \right] = 0 \end{aligned}$$

for every $(z_1, z_2) \in M$, which ends the proof. \square

Remark 6. By working out the above differential equations (i), ..., and (v) (cf. Lemma 7 in Appendix A.2), it follows that the functions f and P are defined uniquely up to choices of functions q and p , respectively.

4. PROOF OF THEOREM 1

This section is devoted to proving Theorem 1. To do this, we divide the proof into six following claims from Claim 1 to Claim 6.

As a first step we shall establish several equations that will be of use later. Let $H(z_1, z_2) = h_1(z_1, z_2)\frac{\partial}{\partial z_1} + h_2(z_1, z_2)\frac{\partial}{\partial z_2}$ and M be a non-trivial holomorphic vector field and a real non-rotationally symmetric hypersurface, respectively, as in Theorem 1. Then we have the identity

$$(\operatorname{Re} H)\rho(z) = 0, \quad \forall z \in M. \quad (2)$$

Expand h_1 and h_2 into the Taylor series at the origin so that

$$\begin{aligned} h_1(z_1, z_2) &= \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k = \sum_{j=0}^{\infty} z_1^j a_j(z_2); \\ h_2(z_1, z_2) &= \sum_{j,k=0}^{\infty} b_{jk} z_1^j z_2^k = \sum_{j=0}^{\infty} z_1^j b_j(z_2), \end{aligned}$$

where $a_{jk}, b_{jk} \in \mathbb{C}$ and a_j, b_j are holomorphic in a neighborhood of $0 \in \mathbb{C}$ for all $j, k \in \mathbb{N}$. We note that $a_{00} = b_{00} = 0$ since $h_1(0, 0) = h_2(0, 0) = 0$. Moreover, the function (z_2, t) can be written as

$$f(z_2, t) = tQ(z_2, t) = \sum_{j=0}^{\infty} t^{j+1} Q_j(z_2),$$

where Q_j ($j = 1, 2, \dots$) are real analytic in a neighborhood of $0 \in \mathbb{C}$ and $Q(z_2, t) := \sum_{j=0}^{\infty} t^j Q_j(z_2)$.

By a simple computation, we have

$$\begin{aligned} \rho_{z_1}(z_1, z_2) &= \frac{1}{2} + \frac{Q(z_2, \operatorname{Im} z_1)}{2i} + (\operatorname{Im} z_1)Q_{z_1}(z_2, \operatorname{Im} z_1) \\ &= \frac{1}{2} + \frac{Q_0(z_2)}{2i} + \frac{2(\operatorname{Im} z_1)Q_1(z_2)}{2i} + \frac{3(\operatorname{Im} z_1)^2 Q_2(z_2)}{2i} + \dots; \\ \rho_{z_2}(z_1, z_2) &= P'(z_2) + (\operatorname{Im} z_1)Q_{z_2}(z_2, \operatorname{Im} z_1), \end{aligned}$$

and the equation (2) can thus be re-written as

$$\begin{aligned} \operatorname{Re} \left[\left(\frac{1}{2} + \frac{Q(z_2, \operatorname{Im} z_1)}{2i} + (\operatorname{Im} z_1)Q_{z_1}(z_2, \operatorname{Im} z_1) \right) h_1(z_1, z_2) \right. \\ \left. + \left(P'(z_2) + (\operatorname{Im} z_1)Q_{z_2}(z_2, \operatorname{Im} z_1) \right) h_2(z_1, z_2) \right] = 0, \end{aligned} \quad (3)$$

for all $(z_1, z_2) \in M$.

Since $(it - P(z_2) - tQ(z_2, t), z_2) \in M$ for any $t \in \mathbb{R}$ with t small enough, the above equation again admits a new form

$$\begin{aligned} \operatorname{Re} \left[\left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} + \frac{2tQ_1(z_2)}{2i} + \frac{3t^2Q_2(z_2)}{2i} + \dots \right) \times \right. \\ \left. \left(\sum_{j=0}^{\infty} (it - P(z_2) - tQ_0(z_2) - t^2Q_1(z_2) - \dots)^j a_j(z_2) \right) \right. \\ \left. + \left(P'(z_2) + tQ_{0z_2}(z_2) + t^2Q_{1z_2}(z_2) + \dots \right) \times \right. \\ \left. \left(\sum_{m=0}^{\infty} (it - P(z_2) - tQ_0(z_2) - t^2Q_1(z_2) - \dots)^m b_m(z_2) \right) \right] = 0 \end{aligned} \quad (4)$$

for all $z_2 \in \mathbb{C}$ and for all $t \in \mathbb{R}$ with $|z_0| < \epsilon_0$ and $|t| < \delta_0$, where $\epsilon_0 > 0$ and $\delta_0 > 0$ are small enough.

The next step is to demonstrate the following claims. First of all, the following is the first claim, in which its proof only requires the properties (2) and (3) of the function P .

Claim 1. $h_1(0, z_2) \equiv 0$ and $h_2(0, z_2) = i\beta z_2 + \dots$ for some $\beta \in \mathbb{R}^*$.

Proof of the claim. Indeed, it follows from (3) with $t = 0$ that

$$\operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} Q(z_2, 0) \right) h_1(0, z_2) \right] + O(P(z_2)) + O(P'(z_2)) = 0. \quad (5)$$

Since $\nu_0(P) = \nu_0(P') = +\infty$, it follows from the equation (5) that

$$\operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} Q(z_2, 0) \right) h_1(0, z_2) \right] = 0.$$

Notice that $h_1(0, 0) = 0$ and $Q(0, 0) = 0$. So, it is easy to show that the above equation implies that $h_1(0, z_2) \equiv 0$.

Notice that we may choose $t = \alpha P(z_2)$ in (3) (with $\alpha \in \mathbb{R}$ to be chosen later on). Then one gets

$$\begin{aligned} & \operatorname{Re} \left[\left(\frac{1}{2} + \frac{1}{2i} Q(z_2, \alpha P(z_2)) + \alpha P(z_2) Q_{z_1}(z_2, \alpha P(z_2)) \right) \times \right. \\ & \quad h_1 \left(i\alpha P(z_2) - P(z_2) - \alpha P(z_2) Q(z_2, \alpha P(z_2)), z_2 \right) \\ & \quad \left. + \left(P'(z_2) + \alpha P(z_2) Q_{z_2}(z_2, \alpha P(z_2)) \right) \times \right. \\ & \quad \left. h_2 \left(i\alpha P(z_2) - P(z_2) - \alpha P(z_2) Q(z_2, \alpha P(z_2)), z_2 \right) \right] = 0 \end{aligned} \quad (6)$$

for all $z_2 \in \Delta_{\epsilon_0}$.

We note that if $h_2 \equiv 0$, then (3) shows that $h_1 \equiv 0$. So we may assume that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$. Let j_0 be the smallest integer number such that $a_{j_0 k} \neq 0$ for some integer number k . Then let k_0 be the smallest integer number such that $a_{j_0 k_0} \neq 0$. Similarly, let m_0 be the smallest integer number such that $b_{m_0 n} \neq 0$ for some integer number n . Then let n_0 be the smallest integer number such that $a_{m_0 n_0} \neq 0$. Note that $j_0 \geq 1$ since $h_1(0, z_2) \equiv 0$.

Since $P(z_2) = o(|z_2|^{n_0})$, it follows from (6) that

$$\begin{aligned} & \operatorname{Re} \left[\frac{1}{2} a_{j_0 k_0} (i\alpha - 1)^{j_0} (P(z_2))^{j_0} z_2^{k_0} + b_{m_0 n_0} (i\alpha - 1)^{m_0} (z_2^{n_0} + o(|z_2|^{n_0})) \right. \\ & \quad \left. \times (P(z_2))^{m_0} \left(P'(z_2) + \alpha P(z_2) Q_{z_2}(z_2, \alpha P(z_2)) \right) \right] = o(P(z_2)^{j_0} |z_2|^{k_0}), \end{aligned} \quad (7)$$

for all $|z_2| < \epsilon_0$ and for all $\alpha \in \mathbb{R}$. We note that in the case $k_0 = 0$ and $\operatorname{Re}(a_{j_0 0}) = 0$, α can be chosen in such a way that $\operatorname{Re}((i\alpha - 1)^{j_0} a_{j_0 0}) \neq 0$. Then the above equation yields that $j_0 > m_0$. We conclude from [9, Lemma 3] that $m_0 = 0, n_0 = 1$, and $b_{0,1} = i\beta z_2$ for some $\beta \in \mathbb{R}^*$. Therefore, the claim is proved. \square

Now by a change of variables as in Lemma 4, without loss of generality we may assume that $b_0(z_2) = i\beta z_2$. Moreover, we have the following claim.

Claim 2. We have that $a_1(z_2) = \beta \sum_{n=1}^{\infty} a_n z_2^n \neq 0$ and

$$Q_0(z_2) = \tan(R(z_2));$$

$$P(z_2) = \exp \left[p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log |\cos(R(z_2))| + v(z_2) \right]$$

for all $z_2 \in \Delta_{\epsilon_0}^*$, where $R(z_2) = q(|z_2|) - \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n\right)$ for all $z_2 \in \Delta_{\epsilon_0}^*$, v is a \mathcal{C}^∞ -smooth function on Δ_{ϵ_0} with $\nu_0(v) = +\infty$, and q, p are \mathcal{C}^∞ -smooth functions on $(0, \epsilon_0)$ and are chosen so that R is real analytic in Δ_{ϵ_0} and that P is \mathcal{C}^∞ -smooth in Δ_{ϵ_0} with $\nu_0(P) = +\infty$.

Proof of the claim. First of all, taking $\frac{\partial}{\partial t}$ of both sides of the equation (4) at $t = 0$, we obtain that

$$\begin{aligned} \operatorname{Re}\Big\{P'(z_2)(i - Q_0(z_2))\Big[& b_1(z_2) + 2(-P(z_2))b_2(z_2) + \cdots \\ & + m(-P(z_2))^{m-1}b_m(z_2) + \cdots\Big] \\ & + \frac{i}{2}\left(1 + Q_0^2(z_2)\right)\Big[a_1(z_2) + 2(-P(z_2))a_2(z_2) + \cdots \\ & + m(-P(z_2))^{m-1}a_m(z_2) + \cdots\Big] \\ & + Q_{0z_2}\Big[i\beta z_2 + (-P(z_2))b_1(z_2) + \cdots + (-P(z_2))^m b_m(z_2) + \cdots\Big] \\ & + \frac{Q_1(z_2)}{i}\Big[(-P(z_2))a_1(z_2) + (-P(z_2))^2 a_2(z_2) + \cdots \\ & + (-P(z_2))^m a_m(z_2) + \cdots\Big]\Big\} = 0 \end{aligned} \quad (8)$$

for all $z_2 \in \Delta_{\epsilon_0}$. Since Q_0 is real-analytic and $\nu_0(P) = \nu_0(P') = 0$, one gets

$$\operatorname{Re}\left[i\beta z_2 Q_{0z_2}(z_2) + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\right)(i - Q_0(z_2))a_1(z_2)\right] \equiv 0, \quad (9)$$

or equivalently

$$\operatorname{Re}\left[2i\beta z_2 Q_{0z_2}(z_2) + ia_1(z_2)\left(1 + Q_0^2(z_2)\right)\right] \equiv 0 \quad (10)$$

on Δ_{ϵ_0} . We note that the equation (10) shows that $\operatorname{Re}(ia_1(0)) = 0$.

Therefore, the solution Q_0 of Eq. (10) has the form as in the claim (see Lemma 7 in Appendix A.2). In addition, since the real hypersurface M is not rotationally symmetric, by [9, Theorem 3] mentioned as in the introduction, Q_0 must contain a monomial term z_2^k for some positive integer k . Consequently, we have in fact that $a_1 \not\equiv 0$.

Next, it follows from (4) with $t = 0$ that

$$\operatorname{Re}\left[-\left(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\right)a_1(z_2)P(z_2) + i\beta z_2 P'(z_2)\right] + O(P(z_2)^2) + O(P'(z_2)P(z_2)) = 0, \quad (11)$$

or equivalently

$$2\operatorname{Re}\left(i\beta z_2 \frac{P_{z_2}(z_2)}{P(z_2)}\right) = \operatorname{Re}(a_1(z_2)) + Q_0(z_2)\operatorname{Re}\left(\frac{a_1(z_2)}{i}\right) + O(P(z_2)) + O(P'(z_2)) \quad (12)$$

for every $z_2 \in \Delta_{\epsilon_0}$. By [9, Lemma 1], it follows from Eq. (12) that $\operatorname{Re}(a_1(0)) = 0$, which, together with the above-mentioned fact that $\operatorname{Re}(ia_1(0)) = 0$, shows that $a_1(0) = 0$.

Now the solution P of the Eq. (12) has the form as claimed (see Lemma 7 in Appendix A.2). Hence, the proof is complete. \square

We now observe that $\limsup_{r \rightarrow 0^+} |rp'(r)| = +\infty$, for otherwise one gets $|p(r)| \lesssim |\log(r)|$ for every $0 < r < \epsilon_0$, and thus P does not vanish to infinite order at 0. Furthermore, a direct calculation shows that

$$z_2 \frac{P_{z_2}(z_2)}{P(z_2)} = \frac{1}{2} |z_2| p'(|z_2|) + g(z_2) \quad (13)$$

for all $|z_2| < \epsilon_0$, where g is a \mathcal{C}^∞ -smooth function.

Claim 3. $b_1 \equiv 0$ on Δ_{ϵ_0} .

Proof of the claim. To obtain a contradiction, we suppose that $b_1 \not\equiv 0$, it follows from (8) and (9) that

$$\begin{aligned} \operatorname{Re} \left\{ \left((i - Q_0(z_2)) b_1(z_2) \right) \frac{P'(z_2)}{P(z_2)} - i a_2(z_2) (1 + Q_0^2(z_2)) \right. \\ \left. - Q_{0z_2}(z_2) b_1(z_2) - \frac{Q_1(z_2)}{i} a_1(z_2) + O(P(z_2)) + O(P'(z_2)) \right\} \equiv 0 \end{aligned} \quad (14)$$

on Δ_{ϵ_0} . We will show that $b_1(z_2) = \tilde{\beta} z_2 + \dots$ for some $\tilde{\beta} \in \mathbb{R}^*$. To prove this, we consider the following cases.

Case 1. $b_1(0) \neq 0$. In this case, let $\gamma : (-1, 1) \rightarrow \Delta_{\epsilon_0} \subset \mathbb{C}$ be a \mathcal{C}^∞ -smooth curve such that $\gamma'(t) = (i - Q_0(\gamma(t))) b_1(\gamma(t))$ for all $|t| < 1$ and $\gamma(0) = 0$. It follows from (14) that $\operatorname{Re} \left((i - Q_0(z_2)) b_1(z_2) P_{z_2}(z_2) / P(z_2) \right)$ is bounded on $\Delta_{\epsilon_0}^*$ and thus

$$\frac{d}{dt} \log P(\gamma(t)) = 2 \operatorname{Re} \left(\gamma'(t) P_{z_2}(\gamma(t)) / P_{z_2}(\gamma(t)) \right)$$

is also bounded on $(-1, 1)$. This implies that $\log P(\gamma(t)) = O(t)$, which contradicts the fact that $P(\gamma(t)) \rightarrow 0$ as $t \rightarrow 0$. Therefore, we conclude that $b_1(0) = 0$.

Case 2. $b_1'(0) \notin \mathbb{R}^*$. It follows from (13) and (14) that

$$\operatorname{Re} \left((i - Q_0(z_2)) \tilde{b}_1(z_2) |z_2| p'(|z_2|) \right) - \tilde{g}(z_2) = 0$$

for all $z_2 \in \Delta_{\epsilon_0}$, where $\tilde{g}(z_2)$ is a \mathcal{C}^∞ -smooth real-valued function defined on Δ_{ϵ_0} with $l := \nu_0(\tilde{g}(z_2)) \geq k$ and $\tilde{b}_1(z_2) := b_1(z_2)/z_2$ if $z_2 \neq 0$ and $\tilde{b}_1(0) = b_1'(0)$. Since P satisfies the property (4), that is, there is no positive integer n such that $\log P(z_2) \approx \frac{1}{|z_2|^n}$, and $b_1'(0) \notin \mathbb{R}^*$, the function $\operatorname{Re} \left((i - Q_0(z_2)) \tilde{b}_1(z_2) |z_2| p'(|z_2|) \right)$ cannot extend to be \mathcal{C}^∞ -smooth in Δ_{ϵ_0} , which is a contradiction.

Therefore, we obtain that $b_1(z_2) = \tilde{\beta} z_2 + \dots$ for some $\tilde{\beta} \in \mathbb{R}^*$. Furthermore, from (12) and (14) we have that

$$\begin{aligned} \operatorname{Re} \left\{ \left((i - Q_0(z_2)) b_1(z_2) - i \tilde{\beta} z_2 \right) \frac{P'(z_2)}{P(z_2)} \right\} - (1 + Q_0^2(z_2)) \operatorname{Re}(i a_2(z_2)) \\ - \operatorname{Re}(Q_{0z_2}(z_2) b_1(z_2)) - \operatorname{Re} \left(\frac{Q_1(z_2)}{i} a_1(z_2) \right) \\ - \frac{\tilde{\beta}}{2\beta} \left(\operatorname{Re}(a_1(z_2)) + Q_0(z_2) \operatorname{Re} \left(\frac{a_1(z_2)}{i} \right) \right) + O(P(z_2)) + O(P'(z_2)) \equiv 0 \end{aligned} \quad (15)$$

on $\Delta_{\epsilon_0}^*$. Let us denote by $c(z_2)$ the real analytic function on Δ_{ϵ_0} defined by

$$c(z_2) := \frac{(i - Q_0(z_2)) b_1(z_2) - i \tilde{\beta} z_2}{z_2}$$

for all $z_2 \in \Delta_{\epsilon_0}^*$. It is easy to see that $\operatorname{Re}(c(z_2)) \neq 0$. Moreover, by (13) and (15) the function $\operatorname{Re}(c(z_2))|z_2|p'(|z_2|)$ extends to be \mathcal{C}^∞ -smooth in Δ_{ϵ_0} . However, it is again impossible since P satisfies the property (4). Thus, we conclude that $b_1 \equiv 0$. \square

Claim 4. $a_2(z_2) \equiv Q_1(0)a_1(z_2)$ and $Q_1(z_2) \equiv Q_1(0)(1 + Q_0^2(z_2))$ on Δ_{ϵ_0} .

Proof of the claim. Since $b_1 \equiv 0$ (cf. Claim 3), by (14) and note that Q_0, Q_1 are real analytic, and $P(z_2), P'(z_2)$ vanish to infinite order at 0, one has

$$\operatorname{Re}\left[i\left(1 + Q_0^2(z_2)\right)a_2(z_2) - iQ_1(z_2)a_1(z_2)\right] \equiv 0 \quad (16)$$

on Δ_{ϵ_0} .

On the other hand, taking $\frac{\partial^2}{\partial t^2}$ of both sides of Eq. (4) at $t = 0$, we have that

$$\begin{aligned} & \operatorname{Re}\left\{\frac{3Q_2(z_2)}{2i}\left(-P(z_2)a_1(z_2) + P(z_2)^2a_2(z_2) + \cdots (-P(z_2))^{m-1}a_{m-1}(z_2) \cdots\right)\right. \\ & + \frac{Q_1(z_2)}{i}(i - Q_0(z_2))\left(a_1(z_2) - 2P(z_2)a_2(z_2) + \cdots\right. \\ & + m(-P(z_2))^{m-1}a_m(z_2) + \cdots) + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\right) \\ & \times \left(-Q_1(z_2)a_1(z_2) + \left[(i - Q_0(z_2))^2 + 2P(z_2)Q_1(z_2)\right]a_2(z_2) + \cdots\right. \\ & + \left[\frac{(m+1)m}{2}(-P(z_2))^{m-1}(i - Q_0(z_2))^2 - (m+1)(-P(z_2))^mQ_1(z_2)\right]a_{m+1}(z_2) \\ & + \cdots) + (Q_0)_{z_2}(z_2)(i - Q_0(z_2))\left(b_1(z_2) - 2P(z_2)b_2(z_2) + \cdots\right. \\ & + m(-P(z_2))^{m-1}b_m(z_2) + \cdots) \\ & + (Q_1)_{z_2}(z_2)\left(i\beta z_2 - P(z_2)b_1(z_2) + \cdots + (-P(z_2))^mb_m(z_2) + \cdots\right) \\ & + P'(z_2)\left(-Q_1(z_2)b_1(z_2) + \left[(i - Q_0(z_2))^2 + 2P(z_2)Q_1(z_2)\right]b_2(z_2) + \cdots\right. \\ & + \left[\frac{m(m-1)}{2}(-P(z_2))^{m-2}(i - Q_0(z_2))^2 - m(-P(z_2))^{m-1}Q_1(z_2)\right]b_m(z_2) \\ & \left. + \cdots\right)\left.\right\} \equiv 0 \text{ on } \Delta_{\epsilon_0}. \end{aligned} \quad (17)$$

Since Q_0, Q_1 are real analytic, $\nu_0(P) = \nu_0(P') = +\infty$, and $b_1 \equiv 0$, we deduce that

$$\begin{aligned} & \operatorname{Re}\left\{i\beta z_2(Q_1)_{z_2}(z_2) + \frac{Q_1(z_2)}{i}(i - Q_0(z_2))a_1(z_2) + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\right)\right. \\ & \times \left(-Q_1(z_2)a_1(z_2) + (i - Q_0(z_2))^2a_2(z_2)\right)\left.\right\} \equiv 0 \end{aligned} \quad (18)$$

on Δ_{ϵ_0} . This equation implies that $\operatorname{Re}(a_2(0)) = 0$. Moreover, Eq. (16) shows that $\operatorname{Re}(ia_2(0)) = 0$. Thus $a_2(0) = 0$.

Now the equations (10), (16), and (18) yield the proof of the claim (see Lemma 8 in Appendix A.2). \square

Claim 5. We have that $a_m(z_2) \equiv \frac{2^{m-1}}{m!}Q_1^{m-1}(0)a_1(z_2)$ and $b_{m-1} \equiv 0$ on Δ_{ϵ_0} for all $m \geq 2$.

Proof of the claim. We shall prove the claim by induction on m . For $m = 2$, it follows from Claim 4 and Claim 3 that $a_2 \equiv Q_1(0)a_1(z_2)$ and $b_1 \equiv 0$, respectively. Suppose that $a_2(z_2) \equiv Q_1(0)a_1(z_2), \dots, a_m(z_2) \equiv \frac{2^{m-1}}{m!}Q_1^{m-1}(0)a_1(z_2)$, $b_1 \equiv \dots \equiv b_{m-1} \equiv 0$ for $m \geq 2$. We will show that $b_m \equiv 0$ and $a_{m+1}(z_2) \equiv \frac{2^m}{(m+1)!}Q_1^m(0)a_1(z_2)$.

Indeed, by (8) we have

$$\begin{aligned} \operatorname{Re} \left\{ (-1)^{m-1}m(i - Q_0(z_2))b_m(z_2) \frac{P'(z_2)}{P(z_2)} + (-1)^m(m+1) \frac{i}{2} (1 + Q_0^2(z_2))a_{m+1}(z_2) \right. \\ \left. + (-1)^mb_m(z_2)Q_{0z_2}(z_2) + (-1)^m \frac{Q_1(z_2)}{i} a_m(z_2) + O(P(z_2)) + O(P'(z_2)) \right\} \equiv 0 \end{aligned} \quad (19)$$

on Δ_{ϵ_0} .

Repeating the argument as in the proof of Claim 3, we deduce that $b_m \equiv 0$. Thus we obtain that

$$\operatorname{Re} \left\{ (-1)^m(m+1) \frac{i}{2} (1 + Q_0^2(z_2))a_{m+1}(z_2) + (-1)^m \frac{Q_1(z_2)}{i} a_m(z_2) \right\} \equiv 0. \quad (20)$$

Δ_{ϵ_0} . Consequently, one has $\operatorname{Re}(ia_{m+1}(0)) = 0$.

On the other hand, since Q_0, Q_1, Q_2 are real analytic, $\nu_0(P) = \nu_0(P') = +\infty$, and $b_1 \equiv \dots \equiv b_m \equiv 0$, from (17) we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{3Q_2(z_2)}{2i} a_{m-1}(z_2) + \frac{Q_1(z_2)}{i} (i - Q_0(z_2))a_m(z_2) \right. \\ \left. + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \left(\frac{m(m+1)}{2} (i - Q_0(z_2))^2 a_{m+1}(z_2) - mQ_1(z_2)a_m(z_2) \right) \right\} \equiv 0 \end{aligned} \quad (21)$$

on Δ_{ϵ_0} . This implies that $\operatorname{Re}(a_{m+1}(0)) = 0$, which, together with $\operatorname{Re}(ia_{m+1}(0)) = 0$ as above, indicates that $a_{m+1}(0) = 0$.

Furthermore, since $Q_1(z_2) \equiv Q_1(0) \left(1 + Q_0^2(z_2) \right)$ (cf. Claim 4), we conclude from (20) that

$$a_{m+1}(z_2) \equiv \frac{2}{m+1} Q_1(0)a_m(z_2) \equiv \dots \equiv \frac{2^m}{(m+1)!} Q_1^m(0)a_1(z_2),$$

as claimed. \square

Claim 6. We have that

(a)

$$f(z_2, t) = \begin{cases} -\frac{1}{2Q_1(0)} \log \left| \frac{\cos(R(z_2) + 2Q_1(0)t)}{\cos(R(z_2))} \right| & \text{if } Q_1(0) \neq 0 \\ \tan(R(z_2))t & \text{if } Q_1(0) = 0 \end{cases}$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$, where R is given in Claim 5.

(b)

$$P(z_2) = \begin{cases} \frac{1}{2Q_1(0)} \log \left[1 + 2Q_1(0)P_1(z_2) \right] & \text{if } Q_1(0) \neq 0 \\ P_1(z_2) & \text{if } Q_1(0) = 0 \end{cases}$$

for all $z_2 \in \Delta_{\epsilon_0}$, where

$$P_1(z_2) = \exp \left(p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log \left| \cos(R(z_2)) \right| \right)$$

for all $z_2 \in \Delta_{\epsilon_0}^*$ and $P_1(0) = 0$, where p, q are the functions given in Claim 2.

Proof of the claim . By Claim 5, it is easy to check that $h_1(z_1, z_2) = z_1 a_1(z_2)$ if $Q_1(0) = 0$ and

$$h_1(z_1, z_2) = \frac{1}{2Q_1(0)} \left[\exp \left(2Q_1(0)z_1 \right) - 1 \right] a_1(z_2)$$

if $Q_1(0) \neq 0$ and $h_2(z_1, z_2) = i\beta z_2$.

Now we divide the proof into the two following cases.

Case A. $Q_1(0) = 0$. From Eq. (4) we have that

$$\begin{aligned} & \operatorname{Re} \left\{ \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} + \frac{2tQ_1(z_2)}{2i} + \frac{3t^2Q_2(z_2)}{2i} + \dots \right) \right. \\ & \quad \times \left(it - P(z_2) - tQ_0(z_2) - t^2Q_1(z_2) - \dots \right) a_1(z_2) \\ & \quad \left. + \left(P'(z_2) + tQ_{0z_2}(z_2) + t^2Q_{1z_2}(z_2) + \dots \right) i\beta z_2 \right\} = 0 \end{aligned} \quad (22)$$

for all $z_2 \in \mathbb{C}$ and for all $t \in \mathbb{R}$ with $|z_0| < \epsilon_0$ and $|t| < \delta_0$. Then Eq. (22) with $t = 0$ implies easily that

$$\operatorname{Re} \left\{ i\beta z_2 P'(z_2) - \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) P(z_2) a_1(z_2) \right\} \equiv 0 \quad (23)$$

on Δ_{ϵ_0} . Therefore, by Lemma 7 in Appendix A.2 the function $P(z_2) \equiv P_1(z_2)$, as desired.

Now by Claim 4, it follows that $Q_1 \equiv 0$, and thus taking $\frac{\partial^2}{\partial t^2}$ of both sides of (22) at $t = 0$, we obtain that

$$\operatorname{Re} \left(\frac{3Q_2(z_2)}{2i} (-P(z_2)) a_1(z_2) \right) \equiv 0$$

on Δ_{ϵ_0} . This implies that $Q_2 \equiv 0$. Taking $\frac{\partial^m}{\partial t^m}$ of both sides of (22) at $t = 0$ for $m = 3, \dots$, we obtain, by induction on m , that $Q_m \equiv 0$ for all $m \geq 1$. Therefore, from Eq. (22) and Eq. (23) we have

$$\operatorname{Re} \left[2i\beta z_2 Q_{0z_2}(z_2) + ia_1(z_2) \left(1 + Q_0^2(z_2) \right) \right] \equiv 0$$

on Δ_{ϵ_0} . Hence, the solution $Q_0(z_2) = \tan(R(z_2))$ for all $z_2 \in \Delta_{\epsilon_0}$, where R is given in the claim (see Lemma 7 in Appendix A.2), and hence $f(z_2, t) = Q_0(z_2)t = \tan(R(z_2))t$ for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$, as claimed.

Case B. $Q_1(0) \neq 0$. In this case, it follows from (3) that

$$\begin{aligned} & \operatorname{Re} \left\{ \left(\frac{1}{2} + \frac{f_t(z_2, t)}{2i} \right) \frac{1}{2Q_1(0)} \left[\exp \left(2Q_1(0)(it - P(z_2) - f(z_2, t)) \right) - 1 \right] a_1(z_2) \right. \\ & \quad \left. + \left(P'(z_2) + f_{z_2}(z_2, t) \right) i\beta z_2 \right\} = 0 \end{aligned} \quad (24)$$

for all $z_2 \in \mathbb{C}$ and for all $t \in \mathbb{R}$ with $|z_0| < \epsilon_0$ and $|t| < \delta_0$.

Since the function $f(z_2, t) = \sum_{n=1}^{\infty} Q_{n-1}(z_2)t^n$ is real analytic in a neighborhood of $0 \in \mathbb{C} \times \mathbb{R}$, by (24) we obtain the following.

- (i) $\left(i + f_t(z_2, t) \right) \exp \left(2Q_1(0)(it - f(z_2, t)) \right) = i + f_t(z_2, 0) ;$
- (ii) $\operatorname{Re} \left[4iQ_1(0)\beta z_2 f_{z_2}(z_2, t) + \left(f_t(z_2, t) - \tan(R(z_2)) \right) ia_1(z_2) \right] = 0;$

$$(iii) \operatorname{Re}\left(i\beta z_2 P'(z_2)\right) = -\frac{\exp\left(-2Q_1(0)P(z_2)\right)-1}{2Q_1(0)} \operatorname{Re}\left[\left(\frac{1}{2} + \frac{Q_0(z_2)}{2i}\right)a_1(z_2)\right]$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$.

Now by (i), it follows from Lemma 9 in Appendix A.2 with $\alpha = 2Q_1(0)$ that

$$f(z_2, t) = \begin{cases} -\frac{1}{2Q_1(0)} \log \left| \frac{\cos(R(z_2) + 2Q_1(0)t)}{\cos(R(z_2))} \right| & \text{if } Q_1(0) \neq 0 \\ \tan(R(z_2))t & \text{if } Q_1(0) = 0 \end{cases}$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$. We note that

$$2\operatorname{Re}\left(i\beta z_2 R_{z_2}(z_2)\right) = -\operatorname{Re}(ia_1(z_2))$$

for all $z_2 \in \Delta_{\epsilon_0}$. Hence, by Corollary 1 in Appendix A.2 we conclude that Eq. (ii) automatically holds. Finally, by Eq. (iii) and Lemma 7 in Appendix A.2 with $\alpha = 2Q_1(0)$, we conclude that the function $P(z_2)$ has the form as in the claim.

Altogether, the claim is proved. \square

In conclusion, Claim 1, Claim 2, ..., and Claim 6 complete the proof of Theorem 1 (modulo Lemma 7, Lemma 8, Lemma 9, and Corollary 1 which we prove in Appendix A.2). \square

5. PROOF OF THEOREM 2

This section is entirely devoted to the proof of Theorem 2. Let $M = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_1 + P(z_2) + (\operatorname{Im} z_1) Q(z_2, \operatorname{Im} z_1) = 0\}$ be the real hypersurface germ at 0 described in the hypothesis of Theorem 2. Our present goal is to show that there is no non-trivial holomorphic vector field vanishing at the origin and tangent to M .

For the sake of smooth exposition, we shall present the proof in two subsections. In Subsection 5.1, several technical lemmas are introduced. Then the proof of Theorem 2 is presented in Subsection 5.2. Throughout what follows, for $r > 0$ denote by $\tilde{\Delta}_r := \{z_2 \in \Delta_r : P(z_2) \neq 0\}$.

5.1. Technical lemmas. Since P satisfies the condition (I), it is not hard to show the following two lemmas.

Lemma 5. *If a, b are complex numbers and if g_1, g_2 are smooth functions defined on the disc Δ_{ϵ_0} with sufficiently small radius $\epsilon_0 > 0$ satisfying*

- (i) $g_1(z) = O(|z|^{\ell+1})$ and $g_2(z) = o(|z|^m)$;
- (ii) $\operatorname{Re}\left[az^m + \frac{b}{P^n(z)}\left(z^{\ell+1}\frac{P'(z)}{P(z)} + g_1(z)\right)\right] = g_2(z)$

for every $z \in \tilde{\Delta}_{\epsilon_0}$ and for any non-negative integers ℓ, m , then $a = b = 0$.

Proof. The proof follows easily from the condition (I.1). \square

Lemma 6. *Let P be a function defined on Δ_{ϵ_0} ($\epsilon_0 > 0$) satisfying the condition (I). Let $B \in \mathbb{C}^*$ and $m \in \mathbb{N}^*$. Then there exists $\alpha \in \mathbb{R}$ such that*

$$\limsup_{\tilde{\Delta}_{\epsilon_0} \ni z \rightarrow 0} |\operatorname{Re}(B(i\alpha - 1)^m P'(z)/P(z))| = +\infty.$$

Proof. Since P satisfies the condition (I.2), there exists a sequence $\{z_k\} \subset \tilde{\Delta}_{\epsilon_0}$ converging to 0 such that $\lim_{k \rightarrow \infty} P'(z_k)/P(z_k) = \infty$. We can write

$$\begin{aligned} BP'(z_k)/P(z_k) &= a_k + ib_k, \quad k = 1, 2, \dots; \\ (i\alpha - 1)^m &= a(\alpha) + ib(\alpha). \end{aligned}$$

We note that $|a_k| + |b_k| \rightarrow +\infty$ as $k \rightarrow \infty$. Therefore, passing to a subsequence if necessary, we only consider two following cases.

Case 1. $\lim_{k \rightarrow \infty} a_k = \infty$ and $|\frac{b_k}{a_k}| \lesssim 1$. Since $a(\alpha) \rightarrow 1$ and $b(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0$, if α is small enough then

$$\begin{aligned} \operatorname{Re}\left(B(i\alpha - 1)^m P'(z_k)/P(z_k)\right) &= a(\alpha)a_k - b(\alpha)b_k \\ &= a_k\left(a(\alpha) - b(\alpha)\frac{b_k}{a_k}\right) \rightarrow \infty \end{aligned}$$

as $k \rightarrow \infty$.

Case 2. $\lim_{k \rightarrow \infty} b_k = \infty$ and $\lim_{k \rightarrow \infty} |\frac{a_k}{b_k}| = 0$. Fix a real number α such that $b(\alpha) \neq 0$. Then we have

$$\begin{aligned} \operatorname{Re}\left(B(i\alpha - 1)^m P'(z_k)/P(z_k)\right) &= a(\alpha)a_k - b(\alpha)b_k \\ &= b_k\left(a(\alpha)\frac{a_k}{b_k} - b(\alpha)\right) \rightarrow \infty \end{aligned}$$

as $k \rightarrow \infty$. Hence, the proof is complete. \square

5.2. Tangential holomorphic vector fields: Proof of Theorem 2. The CR hypersurface germ $(M, 0)$ at the origin in \mathbb{C}^2 under consideration is defined by the equation $\rho(z_1, z_2) = 0$, where

$$\rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) + (\operatorname{Im} z_1) Q(z_2, \operatorname{Im} z_1) = 0,$$

where P, Q are \mathcal{C}^∞ -smooth functions satisfying the three conditions specified in the hypothesis of Theorem 2, stated in Section 1. Recall that P vanishes to infinite order at $z_2 = 0$ in particular.

Then we consider a holomorphic vector field $H = h_1(z_1, z_2)\frac{\partial}{\partial z_1} + h_2(z_1, z_2)\frac{\partial}{\partial z_2}$ defined on a neighborhood of the origin. We only consider H that is tangent to M , which means that they satisfy the identity

$$(\operatorname{Re} H)\rho(z) = 0, \quad \forall z \in M. \quad (25)$$

The goal is to show that $H \equiv 0$. Indeed, striving for a contradiction, suppose that $H \not\equiv 0$. We notice that if $h_2 \equiv 0$ then (25) shows that $h_1 \equiv 0$. So, without loss of generality we may assume that $h_1 \not\equiv 0$ and $h_2 \not\equiv 0$.

Expand h_1 and h_2 into the Taylor series at the origin so that

$$h_1(z_1, z_2) = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \text{ and } h_2(z_1, z_2) = \sum_{j,k=0}^{\infty} b_{jk} z_1^j z_2^k,$$

where $a_{jk}, b_{jk} \in \mathbb{C}$. We note that $a_{00} = b_{00} = 0$ since $h_1(0, 0) = h_2(0, 0) = 0$.

Next, let us denote by j_0 the smallest integer number such that $a_{j_0 k} \neq 0$ for some integer number k . Then let k_0 be the smallest integer number such that $a_{j_0 k_0} \neq 0$. Similarly, let m_0 be the smallest integer number such that $b_{m_0 n} \neq 0$ for some integer number n . Then denote by n_0 the smallest integer number such that $a_{m_0 n_0} \neq 0$. We note that $j_0 \geq 1$ if $k_0 = 0$ and $m_0 \geq 1$ if $n_0 = 0$.

Following the arguments in the proof of Theorem 1, we obtain that

$$\begin{aligned} \operatorname{Re} \left[\frac{1}{2} a_{j_0 k_0} (i\alpha - 1)^{j_0} (P(z_2))^{j_0} z_2^{k_0} + b_{m_0 n_0} (i\alpha - 1)^{m_0} (z_2^{n_0} + o(|z_2|^{n_0})) (P(z_2))^{m_0} \right. \\ \left. \times \left(P'(z_2) + \alpha P(z_2) Q_{z_2}(z_2, \alpha P(z_2)) \right) \right] = o(P(z_2)^{j_0} |z_2|^{k_0}), \end{aligned} \quad (26)$$

for all $|z_2| < \epsilon_0$ and for any $\alpha \in \mathbb{R}$. We note that in the case $k_0 = 0$ and $\operatorname{Re}(a_{j_0 0}) = 0$, α can be chosen in such a way that $\operatorname{Re}((i\alpha - 1)^{j_0} a_{j_0 0}) \neq 0$. Then the above equation yields that $j_0 > m_0$.

We now divide the argument into two cases as follows.

Case 1. $n_0 \geq 1$. In this case (26) contradicts Lemma 5.

Case 2. $n_0 = 0$. Since P satisfies the condition (I) and $m_0 \geq 1$, by Lemma 6 we can choose a real number α such that

$$\limsup_{\tilde{\Delta}_{\epsilon_0} \ni z_2 \rightarrow 0} |\operatorname{Re}(b_{m_0}(i\alpha - 1)^{m_0} P'(z_2)/P(z_2))| = +\infty,$$

where $\tilde{\Delta}_{\epsilon_0}$ with $\epsilon_0 > 0$ small enough. Therefore, (26) is a contradiction, and thus $h_1 \equiv 0$ on a neighborhood of $(0, 0)$ in \mathbb{C}^2 . Since $h_1 \equiv 0$, it follows from (3) with $t = 0$ that

$$\operatorname{Re} \left[\sum_{m,n=0}^{\infty} b_{mn} z_2^n P'(z_2) \right] = 0,$$

for every z_2 satisfying $|z_2| < \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small. Since P satisfies the condition (I.1), we conclude that $b_{mn} = 0$ for every $m \geq 0, n \geq 1$. We now show that $b_{m0} = 0$ for every $m \in \mathbb{N}^*$. Indeed, suppose otherwise. Then let m be the smallest positive integer such that $b_{m0} \neq 0$. It follows from (6) in the proof of Theorem 1 that

$$\operatorname{Re} \left(b_{m0}(i\alpha - 1)^m P'(z_2)/P(z_2) \right)$$

is bounded on $\tilde{\Delta}_{\epsilon_0}$ with $\epsilon_0 > 0$ small enough for any $\alpha \in \mathbb{R}$. By Lemma 6, this is again impossible.

Altogether, the proof of Theorem 2 is complete. \square

APPENDIX A

A.1. Proof of Lemma 2. Let $G : (0, +\infty) \rightarrow \mathbb{R}$ be the piecewise linear function such that $G(a_n - \epsilon_n) = G(b_n + \epsilon_n) = -2n$ and $G(x) = -8$ if $x \geq \frac{9}{40}$, where $a_n = \frac{1}{n+1}(1 + \frac{1}{3n})$, $b_n = \frac{1}{n+1}(1 + \frac{2}{3n})$, and $\epsilon_n = \frac{1}{n^3}$ for every $n \geq 4$.

Let ψ be a \mathcal{C}^∞ -smooth function on \mathbb{R} given by

$$\psi(x) = C \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $C > 0$ is chosen so that $\int_{\mathbb{R}} \psi(x) dx = 1$. For $\epsilon > 0$, set $\psi_\epsilon := \frac{1}{\epsilon} \psi(\frac{x}{\epsilon})$. For $n \geq 4$, let g_n be the \mathcal{C}^∞ -smooth on \mathbb{R} defined by the following convolution

$$g_n(x) := G * \psi_{\epsilon_{n+1}}(x) = \int_{-\infty}^{+\infty} G(y) \psi_{\epsilon_{n+1}}(y - x) dy.$$

Now we show the following.

- (a) $g_n(x) = G(x) = -2n$ if $a_n \leq x \leq b_n$;
- (b) $g_n(x) = G(x) = -2(n+1)$ if $a_{n+1} \leq x \leq b_{n+1}$;
- (c) $|g_n^{(k)}(x)| \leq \frac{2(n+1)\|\psi^{(k)}\|_1}{\epsilon_{n+1}^k}$ if $a_{n+1} \leq x \leq b_n$.

Indeed, for $a_{n+1} \leq x \leq b_n$ we have

$$\begin{aligned} g_n(x) &= \int_{-\infty}^{+\infty} G(y)\psi_{\epsilon_{n+1}}(y-x)dy \\ &= \frac{1}{\epsilon_{n+1}} \int_{-\infty}^{+\infty} G(y)\psi\left(\frac{y-x}{\epsilon_{n+1}}\right)dy \\ &= \int_{-1}^{+1} G(x+t\epsilon_{n+1})\psi(t)dt, \end{aligned}$$

where we use a change of variable $t = \frac{y-x}{\epsilon_{n+1}}$.

If $a_n \leq x \leq b_n$, then $a_n - \epsilon_n < a_n - \epsilon_{n+1} \leq x + t\epsilon_{n+1} \leq b_n + \epsilon_{n+1} < b_n + \epsilon_n$ for all $-1 \leq t \leq 1$. Therefore,

$$g_n(x) = \int_{-1}^{+1} G(x+t\epsilon_{n+1})\psi(t)dt = -2n \int_{-1}^{+1} \psi(t)dt = -2n,$$

which proves (a). Similarly, if $a_{n+1} \leq x \leq b_{n+1}$, then $a_{n+1} - \epsilon_{n+1} \leq x + t\epsilon_{n+1} \leq b_{n+1} + \epsilon_{n+1}$ for every $-1 \leq t \leq 1$. Hence,

$$g_n(x) = \int_{-1}^{+1} G(x+t\epsilon_{n+1})\psi(t)dt = -2(n+1) \int_{-1}^{+1} \psi(t)dt = -2(n+1),$$

which finishes (b). Moreover, we have the following estimate

$$\begin{aligned} |g_n^{(k)}(x)| &= \frac{1}{\epsilon_{n+1}^k} \left| \int_{-\infty}^{+\infty} G(y)\psi^{(k)}\left(\frac{y-x}{\epsilon_{n+1}}\right)dy \right| \\ &= \frac{1}{\epsilon_{n+1}^k} \left| \int_{-1}^{+1} G(x+t\epsilon_{n+1})\psi^{(k)}(t)dt \right| \\ &\leq \frac{1}{\epsilon_{n+1}^k} \int_{-1}^{+1} |G(x+t\epsilon_{n+1})||\psi^{(k)}(t)|dt \\ &\leq \frac{2(n+1)}{\epsilon_{n+1}^k} \int_{-1}^{+1} |\psi^{(k)}(t)|dt \\ &= \frac{2(n+1)\|\psi^{(k)}\|_1}{\epsilon_{n+1}^k} \end{aligned}$$

for $a_{n+1} \leq x \leq b_n$, where we use again a change of variable $t = \frac{x-y}{\epsilon_{n+1}}$ and the last inequality in the previous equation follows from the fact that $|G(y)| \leq 2(n+1)$ for all $a_{n+1} - \epsilon_{n+1} \leq y \leq b_n + \epsilon_n$. So, the assertion (c) is shown.

Now because of properties (a) and (b) the function

$$g(x) = \begin{cases} -8 & \text{if } x \geq \frac{9}{40} \\ g_n(x) & \text{if } a_{n+1} \leq x \leq b_n, \ n = 4, 5, \dots, \end{cases}$$

is well-defined. From the property (c), it is easy to show that $|g^{(k)}(x)| \lesssim \frac{1}{x^{3k+1}}$ for $k = 0, 1, \dots$ and for every $x \in (0, 1)$, where the constant depends only on k . Thus this proves (iii), and the assertions (i) and (ii) are obvious. Hence, the proof is complete. \square

A.2. Several differential equations. In this subsection, we are going to prove several lemmas and a corollary used in the proofs of Theorem 1 and Theorem 3.

Lemma 7. *Let $a_1(z_2) = \beta \sum_{n=1}^{\infty} a_n z_2^n$ be non-zero holomorphic in Δ_{ϵ_0} ($\beta \in \mathbb{R}^*$, $\epsilon_0 > 0$, $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}^*$). Let Q_0, P_1, P be \mathcal{C}^1 -smooth in Δ_{ϵ_0} with P_1, P are positive on $\Delta_{\epsilon_0}^*$ satisfying the following differential equations:*

$$\begin{aligned} \text{(i)} \quad & \operatorname{Re} \left[2i\beta z_2 Q_{0z_2}(z_2) + ia_1(z_2) \left(1 + Q_0^2(z_2) \right) \right] \equiv 0; \\ \text{(ii)} \quad & \operatorname{Re} \left[2i\beta z_2 P_{1z_2}(z_2) - \left(1 + \frac{Q_0(z_2)}{i} \right) a_1(z_2) P_1(z_2) \right] \equiv 0; \\ \text{(iii)} \quad & \operatorname{Re} \left[2i\beta z_2 P_{z_2}(z_2) + \frac{\exp(-\alpha P(z_2)) - 1}{\alpha} \left(1 + \frac{Q_0(z_2)}{i} \right) a_1(z_2) \right] \equiv 0 \end{aligned}$$

on Δ_{ϵ_0} , where $\alpha \in \mathbb{R}^*$. Then we have

$$\begin{aligned} Q_0(z_2) &= \tan \left[q(|z_2|) - \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n \right) \right]; \\ P_1(z_2) &= \exp \left[p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log \left| \cos \left(q(|z_2|) - \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n \right) \right) \right| \right]; \\ P(z_2) &= \frac{1}{\alpha} \log \left[1 + \alpha P_1(z_2) \right] \end{aligned}$$

for all $z_2 \in \Delta_{\epsilon_0}^*$, where q, p are \mathcal{C}^1 -smooth in $(0, \epsilon_0)$ and are chosen so that Q_0, P_1, P are \mathcal{C}^1 -smooth on Δ_{ϵ_0} .

Proof. We first find solutions of the differential equation (i). Indeed, it follows from (i) that

$$\frac{2\operatorname{Re}(i\beta z_2 Q_{0z_2}(z_2))}{1 + Q_0^2(z_2)} = -\operatorname{Re}(ia_1(z_2)) = -\beta \operatorname{Re} \left(i \sum_{n=1}^{\infty} a_n z_2^n \right)$$

for all $z_2 \in \Delta_{\epsilon_0}$. For a fixed positive number $0 < r < \epsilon_0$, set $u(t) := Q_0(re^{it})$ for every $t \in \mathbb{R}$. Then one has $u'(t) = 2\operatorname{Re}(ire^{it} Q_{0z_2}(re^{it}))$, and hence

$$\frac{u'(t)}{1 + u^2(t)} = -\operatorname{Re} \left(i \sum_{n=1}^{\infty} a_n r^n e^{int} \right)$$

for every $t \in \mathbb{R}$.

For any $t \in \mathbb{R}$, by taking the integral \int_0^t of both sides of the above equation we obtain

$$\begin{aligned} \arctan u(t) - \arctan u(0) &= -\operatorname{Re} \left(i \sum_{n=1}^{\infty} a_n r^n \frac{e^{int} - 1}{in} \right) \\ &= -\operatorname{Re} \left(\sum_{n=1}^{\infty} a_n r^n \frac{e^{int} - 1}{n} \right), \end{aligned} \tag{27}$$

and therefore

$$\begin{aligned} u(t) &= \tan \left[\arctan u(0) - \operatorname{Re} \left(\sum_{n=1}^{\infty} a_n r^n \frac{e^{int} - 1}{n} \right) \right] \\ &= \tan \left[\arctan Q_0(r) - \operatorname{Re} \left(\sum_{n=1}^{\infty} a_n r^n \frac{e^{int} - 1}{n} \right) \right]. \end{aligned}$$

Thus any solution of the differential equation (i) has a form as

$$Q_0(z_2) = \tan \left[q(|z_2|) - \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n \right) \right],$$

where q is a \mathcal{C}^1 -smooth real-valued function $[0, \epsilon_0)$, as desired.

Next, we shall solve the differential equation (ii). Indeed, from Eq. (ii) we have

$$2\operatorname{Re} \left(i\beta z_2 \frac{P_1(z_2)}{P_1(z_2)} \right) = \operatorname{Re}(a_1(z_2)) + Q_0(z_2) \operatorname{Re} \left(\frac{a_1(z_2)}{i} \right)$$

for every $z_2 \in \Delta_{\epsilon_0}^*$. In order to find a solution of the above equation, for a fixed positive number $0 < r < \epsilon_0$, again let $u(t) = \log |P(re^{it})|$ for all $t \in \mathbb{R}$. Then one obtains that

$$\begin{aligned} u'(t) &= \operatorname{Re} \left(\sum_{n=1}^{\infty} a_n r^n e^{int} \right) + Q_0(re^{it}) \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{i} r^n e^{int} \right) \\ &= \operatorname{Re} \left(\sum_{n=1}^{\infty} a_n r^n e^{int} \right) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{i} r^n e^{int} \right) \\ &\quad \times \tan \left[q(r) - \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} (r^n e^{int} - r^n) \right) \right] \end{aligned}$$

for all $t \in \mathbb{R}$. Therefore, by taking the integral \int_0^t of both sides of the above equation, any solution of Eq. (ii) has a form as

$$P_1(z_2) = \exp \left[p(|z_2|) + \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{in} z_2^n \right) - \log \left| \cos \left(q(|z_2|) - \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{a_n}{n} z_2^n \right) \right) \right| \right]$$

for all $z_2 \in \Delta_{\epsilon_0}^*$, where p is a \mathcal{C}^1 -smooth function on $(0, \epsilon_0)$ and is chosen so that $P_1(z)$ is \mathcal{C}^1 -smooth on Δ_{ϵ_0} , as desired.

Finally, using the same argument as the above we conclude from Eq. (iii) that

$$P(z_2) = \frac{1}{\alpha} \log \left[1 + P_1(z_2) \right]$$

for all $z_2 \in \Delta_{\epsilon_0}^*$. Hence, the proof is complete. \square

Lemma 8. Suppose that Q_0, Q_1 are real analytic in Δ_{ϵ_0} ($\epsilon_0 > 0$) of $0 \in \mathbb{C}$ with $Q_0(0) = 0$ and a_1, a_2 are holomorphic in Δ_{ϵ_0} with $a_1(0) = a_2(0) = 0$ and $\nu_0(a_1) < +\infty$ satisfying the following equations:

$$\begin{aligned} \text{(a)} \quad & \operatorname{Re} \left[2i\beta z_2 Q_0(z_2) + ia_1(z_2) (1 + Q_0^2(z_2)) \right] \equiv 0; \\ \text{(b)} \quad & \operatorname{Re} \left[i(1 + Q_0^2(z_2)) a_2(z_2) - iQ_1(z_2) a_1(z_2) \right] \equiv 0; \\ \text{(c)} \quad & \operatorname{Re} \left[i\beta z_2 (Q_1)_{z_2}(z_2) + \frac{Q_1(z_2)}{i} (i - Q_0(z_2)) a_1(z_2) + \left(\frac{1}{2} + \frac{Q_0(z_2)}{2i} \right) \right. \\ & \quad \times \left. \left(-Q_1(z_2) a_1(z_2) + (i - Q_0(z_2))^2 a_2(z_2) \right) \right] \equiv 0 \end{aligned}$$

on Δ_{ϵ_0} . Then we obtain that

$$Q_1(z_2) \equiv Q_1(0) \left(1 + Q_0^2(z_2)\right) \text{ and } a_2(z_2) \equiv Q_1(0)a_1(z_2).$$

Proof. The proof will be divided into two following cases.

Case (i). $Q_1(0) = 0$. In this case, we will show that $Q_1 \equiv 0$, and thus $a_2 \equiv 0$. Indeed, suppose that, contrary to our claim, $Q_1 \not\equiv 0$. Then by (b) we get $\nu_0(a_2) = \nu_0(Q_1) + \nu_0(a_1) > \nu_0(Q_1)$, and moreover Q_1 cannot contain non-harmonic terms of degree $\nu_0(Q_1)$. However, it follows from (c) that $\nu_0(Q_1) = \nu_0(a_2)$, which is a contradiction. Therefore, $Q_1 \equiv 0$ and $a_2 \equiv 0$.

Case (ii). $Q_1(0) \neq 0$. Let $\tilde{Q}_1(z_2) := Q_1(z_2) - Q_1(0)$ and $\tilde{a}_2(z_2) := a_2(z_2) - Q_1(0)a_1(z_2)$ for all $z_2 \in \Delta_{\epsilon_0}$. Then the equation (c) is equivalent to

$$\begin{aligned} & \operatorname{Re} \left\{ i\beta z_2 (\tilde{Q}_1)_{z_2}(z_2) + \frac{1}{2}Q_1(z_2)a_1(z_2) - \frac{3}{2i}Q_1(z_2)Q_0(z_2)a_1(z_2) \right. \\ & \quad \left. - \frac{a_2(z_2)}{2} - \frac{i}{2}Q_0(z_2)a_2(z_2) - \frac{1}{2}Q_0^2(z_2)a_2(z_2) - \frac{i}{2}Q_0^3(z_2)a_2(z_2) \right\} \\ &= \operatorname{Re} \left\{ i\beta z_2 (\tilde{Q}_1)_{z_2}(z_2) - \frac{1}{i}Q_1(0)Q_0(z_2)a_1(z_2) - \frac{1}{2}Q_0^2(z_2)Q_1(0)a_1(z_2) \right. \\ & \quad \left. - \frac{i}{2}Q_0^3(z_2)Q_1(0)a_1(z_2) + \tilde{Q}_1(z_2) \left[\frac{a_1(z_2)}{2} - \frac{3}{2i}Q_0(z_2)a_1(z_2) \right] \right. \\ & \quad \left. + \tilde{a}_2(z_2) \left[-\frac{1}{2} - \frac{i}{2}Q_0(z_2) - \frac{1}{2}Q_0^2(z_2) - \frac{i}{2}Q_0^3(z_2) \right] \right\} \equiv 0 \end{aligned} \quad (28)$$

on Δ_{ϵ_0} . Moreover, the equation (b) is equivalent to

$$\operatorname{Re} \left[i \left(1 + Q_0^2(z_2) \right) \tilde{a}_2(z_2) + i \left(Q_0^2(z_2)Q_1(0) - \tilde{Q}_1(z_2) \right) a_1(z_2) \right] \equiv 0,$$

or equivalently

$$\operatorname{Re} \left[i \left(1 + Q_0^2(z_2) \right) \tilde{a}_2(z_2) + iR_1(z_2)a_1(z_2) \right] \equiv 0 \quad (29)$$

on Δ_{ϵ_0} , where $R_1(z_2) := \tilde{Q}_1(z_2) - Q_0^2(z_2)Q_1(0)$, for simplicity. By (a) and by a simple computation, we get

$$\operatorname{Re} \left\{ i\beta z_2 (Q_0^2(z_2))_{z_2} - Q_0(z_2) \frac{a_1(z_2)}{i} - Q_0^3(z_2) \frac{a_1(z_2)}{i} \right\} \equiv 0$$

on Δ_{ϵ_0} . Hence, it follows from the above equation and (28) that

$$\begin{aligned} & \operatorname{Re} \left\{ i\beta z_2 (R_1)_{z_2}(z_2) - \frac{1}{2}Q_0^2(z_2)Q_1(0)a_1(z_2) + \frac{3}{2i}Q_0^3(z_2)Q_1(0)a_1(z_2) \right. \\ & \quad \left. + Q_1(0)Q_0^2(z_2) \left[\frac{a_1(z_2)}{2} - \frac{3}{2i}Q_0(z_2)a_1(z_2) \right] \right. \\ & \quad \left. + R_1(z_2) \left[\frac{a_1(z_2)}{2} - \frac{3}{2i}Q_0(z_2)a_1(z_2) \right] \right. \\ & \quad \left. + \tilde{a}_2(z_2) \left[-\frac{1}{2} - \frac{i}{2}Q_0(z_2) - \frac{1}{2}Q_0^2(z_2) - \frac{i}{2}Q_0^3(z_2) \right] \right\} \\ &= \operatorname{Re} \left\{ i\beta z_2 (R_1)_{z_2}(z_2) + R_1(z_2) \left[\frac{a_1(z_2)}{2} - \frac{3}{2i}Q_0(z_2)a_1(z_2) \right] \right. \\ & \quad \left. + \tilde{a}_2(z_2) \left[-\frac{1}{2} - \frac{i}{2}Q_0(z_2) - \frac{1}{2}Q_0^2(z_2) - \frac{i}{2}Q_0^3(z_2) \right] \right\} \equiv 0 \end{aligned} \quad (30)$$

on Δ_{ϵ_0} .

Finally, since $R_1(0) = 0$, by the same argument as in Case (i) with (b) and (c) replaced by (29) and (30) respectively, we establish that $R_1 \equiv 0$ and $\tilde{a}_2 \equiv 0$. Hence, $a_2(z_2) \equiv Q_1(0)a_1(z_2)$ and $Q_1(z_2) \equiv Q_1(0)(1 + Q_0^2(z_2))$ on Δ_{ϵ_0} , and the proof is thus complete. \square

Lemma 9. *Let $f(z_2, t)$ be a function defined on a neighborhood $U \times I$ of $0 \in \mathbb{C} \times \mathbb{R}$ with $f(z_2, 0) \equiv 0$ such that f , $\frac{\partial f}{\partial t}$, and $\frac{\partial^2 f}{\partial t^2}$ are \mathcal{C}^1 -smooth on $U \times I$ and let $\alpha \in \mathbb{R}$. Then*

$$\left(i + \frac{\partial f}{\partial t}(z_2, t)\right) \exp\left(\alpha(it - f(z_2, t))\right) \equiv i + \frac{\partial f}{\partial t}(z_2, 0) \text{ on } U \times I$$

if and only if

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0 \end{cases}$$

for all $(z_2, t) \in U \times I$, where R is a \mathcal{C}^1 -smooth function on U .

Proof. It is not hard to check that

$$\frac{\partial}{\partial t} \left[\left(i + \frac{\partial f}{\partial t}(z_2, t)\right) \exp\left(\alpha(it - f(z_2, t))\right) \right] \equiv 0 \text{ on } U \times I$$

if and only if

$$\frac{\partial^2 f}{\partial t^2}(z_2, t) \equiv \alpha \left[1 + \left(\frac{\partial f}{\partial t}(z_2, t)\right)^2 \right] \text{ on } U \times I. \quad (31)$$

Moreover, it follows from Eq. (31) that

$$\frac{\partial f}{\partial t}(z_2, t) = \tan(R(z_2) + \alpha t)$$

for all $(z_2, t) \in U \times I$. Hence, the function f has the form as in the lemma. \square

Corollary 1. *Let $\epsilon_0, \beta, \alpha \in \mathbb{R}$ with $\beta \neq 0$ and $\epsilon_0 > 0$. Suppose that $R : \Delta_{\epsilon_0} \rightarrow [-1, 1]$ is \mathcal{C}^1 -smooth satisfying*

$$2\operatorname{Re}\left(i\beta z_2 R_{z_2}(z_2)\right) = -\operatorname{Re}(ia_1(z_2))$$

for all $z_2 \in \Delta_{\epsilon_0}$, where $a_1(z_2)$ is a non-zero holomorphic function defined on Δ_{ϵ_0} . Let $f(z_2, t) : \Delta_{\epsilon_0} \times (-\delta_0, \delta_0) \rightarrow \mathbb{R}$ be a function defined by

$$f(z_2, t) = \begin{cases} -\frac{1}{\alpha} \log \left| \frac{\cos(R(z_2) + \alpha t)}{\cos(R(z_2))} \right| & \text{if } \alpha \neq 0 \\ \tan(R(z_2))t & \text{if } \alpha = 0, \end{cases}$$

where $\delta_0 = \frac{1}{2|\alpha|}$ if $\alpha \neq 0$ and $\delta_0 = +\infty$ if otherwise. Then we have

$$\operatorname{Re}\left[2i\alpha\beta z_2 f_{z_2}(z_2, t) + \left(f_t(z_2, t) - \tan(R(z_2))\right)ia_1(z_2)\right] = 0 \quad (32)$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$.

Proof. By a direct computation we obtain that

$$f_{z_2}(z_2, t) = \begin{cases} \frac{1}{\alpha} \left(\tan(R(z_2) + \alpha t) - \tan(R(z_2)) \right) R_{z_2}(z_2) & \text{if } \alpha \neq 0 \\ (1 + \tan^2(R(z_2))) R_{z_2}(z_2) t & \text{if } \alpha = 0 \end{cases}$$

and

$$f_t(z_2, t) = \begin{cases} \tan(R(z_2) + \alpha t) & \text{if } \alpha \neq 0 \\ \tan(R(z_2)) & \text{if } \alpha = 0 \end{cases}$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$.

If $\alpha = 0$, then (32) is trivial. So, we only consider the case $\alpha \neq 0$. By our assumption, we thus obtain that

$$\begin{aligned} \operatorname{Re} \left[2i\alpha\beta z_2 f_{z_2}(z_2, t) \right] &= \left(\tan(R(z_2) + \alpha t) - \tan(R(z_2)) \right) \operatorname{Re}(-ia_1(z_2)) \\ &= \left(f_t(z_2, t) - \tan(R(z_2)) \right) \operatorname{Re}(-ia_1(z_2)) \end{aligned}$$

for all $(z_2, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$. Therefore, Eq. (32) holds, and thus the proof is complete. \square

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